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Abstract

We investigate here the confluence of singularities of Mathieu differential equation by means of the Lie algebra contraction of the motion group $M(2)$ on the Heisenberg group $H(3)$.

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1 Introduction

In this paper we deal with second order homogenous differential equations

$$P_0(z)y''(z) + P_1(z)y'(z) + P_2(z)y(z) = 0 \tag{1.1}$$

where $P_0(z)$, $P_1(z)$ and $P_2(z)$ are polynomials in the complex z and should have no common factors. The singularities of (1.1) are defined to be the zeros of the polynomial $P_0(z)$. In the literature the understanding of the solutions of (1.1) in the neighborhood of singularities is very crucial. In general, singularities are branching points of at least one particular solution of (1.1). We recall that there are two types of singularities: the regular (Fuchsian) singularities z_k type for which $(z - z_k)P_1(z)/P_0(z)$ and $(z - z_k)^2 P_2(z)/P_0(z)$ are analytical functions in the neighborhood of z_k and the irregular singularities type which are not regular. A regular singularity (with the corresponding Frobenius solution) is elementary if the difference between two Frobenius exponents is equal to $\frac{1}{2}$; otherwise, it is nonelementary [4]. When two singularities collapse in one point we get, by the so called *confluence*, a new differential equation with possibly different types of singularities and with less polynomial parameters.

The principal object of this work consists in studying the confluence of singularities of Mathieu differential equation to the corresponding singularities of the harmonic oscillator differential equation. The solutions of the former will converge to the solutions of the latter, in a sense which will be precised. We use to that purpose the Lie group contraction procedure performed from the motion group $M(2)$ towards the Heisenberg group $H(3)$ (or precisely on the semi-direct product of $H(3)$ with $\mathbb{Z}_2$). M. Andler and D. Manchon, attempting in [2] to develop the pseudo-differential calculus for finite difference operators, have developed the contraction approach at the group representation level using the Kirillov orbit method. The formalism of multiresolution analysis (MRA), developed previously by Mallat [7], gives them the mean to precise the sense of the limit transformation on group representations, by transforming the irreducible representation in the Hilbert space related to the first group $M(2)$ to the irreducible representation of the Hilbert space $L^2(\mathbb{R})$ of the second group $H(3)$.

To make the link between our work and the results of [2], we should emphasize that the solutions of Mathieu and harmonic oscillator differential equations are eigenvectors of two (unbounded) operators acting respectively on the Hilbert spaces of the two above irreducible representations. These operators both come from a specific second order element in the respective enveloping algebras. The confluence of singularities is then interpreted as a contraction procedure.

We will use below the notion of s-rank [12] which characterize either regular and irregular singular points. To introduce them we associate to (1.1) the *symbolic indicial equation*

$$T(z, D) = 0 \tag{1.2}$$

with $T(z, D) := P_0(z)D^2 + P_1(z)D + P_2(z)$ where D is the differentiation operator and z is a formally independent variable. The two solutions ($m = 1, 2$) of (1.2), in the variable D, can be represented in the neighbourhood of finite singularities z_k by the Puiseux series

$$D_m(z_k) = (z - z_k)^{-\mu_{mk}} \sum_{i=0}^{\infty} h_{mi}(z - z_k)^{i/2}, \quad h_{m0}(z_k) \neq 0 \tag{1.3}$$

or by

$$D_m(\infty) = z^{\mu_{m\infty}-2} \sum_{i=0}^{\infty} h_{mi}(\infty)^{-i/2}, \quad h_{m0}(\infty) \neq 0 \tag{1.4}$$

for the singularity at infinity. The s-rank is then defined respectively for finite and infinite irregular singularities by

$$R_{z_k} = \max_{m=1,2}(\mu_{mk}), \quad R_\infty = \max_{m=1,2}(\mu_{m\infty}). \tag{1.5}$$

Irregular singular points for half-integer s-rank are called *ramified* and *unramified* for integer s-rank. The s-rank of a regular singularity is defined in a different way, and turns out to be 1/2 for an elementary regular singularity, and 1 for a non-elementary one [13]. The set $\{R_{z_1}, R_{z_2}, ..., R_\infty\}$ of s-ranks of singular points of equation (1.1) constitutes its s-multisymbol for which its elements number decreases by one in the case of confluence of two singularities. The two corresponding R_{z_k}'s give rise to a new one greater than their maximum. If the new one is equal to the sum of the original ones then the confluence is called strong. Otherwise, it is called weak.

2 The harmonic oscillator differential equation and the Heisenberg group $H(3)$

The Heisenberg group $H(3)$ can be introduced by a multiplicative operation defined, on the three dimensional space $\mathbb{R} \times \mathbb{R} \times \mathbb{R}$, by

$$(a, b, t)(a', b', t') = (a + a', b + b', t + t' + a.b'). \tag{2.1}$$

or can be realized as group of upper triangular matrices

$$\mathbf{h}(a, b, t) = \begin{pmatrix} 1 & a & t \\ 0 & 1 & b \\ 0 & 0 & 1 \end{pmatrix}. \tag{2.2}$$

One can easily show that

$$\mathbf{h}(a, b, t)\mathbf{h}(a', b', t') = \mathbf{h}(a + a', b + b', t + t' + a.b'). \tag{2.3}$$

The corresponding Lie algebra, noted $h(3)$, is the three dimensional vector space V generated, in the above realization (2.2), by the following matrices

$$P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, Q = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } E = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{2.4}$$

which verify the commutation relations

$$[P, Q] = E, \quad [E, P] = [E, Q] = 0. \tag{2.5}$$

Further, the Heisenberg group $H(3)$ admits unitary representations that are introduced on the Hilbert space $L^2(\mathbb{R})$ of complex functions endowed with the scalar product

$$\langle f_1, f_2 \rangle = \int_{-\infty}^{\infty} f_1(x)\overline{f_2(x)}dx. \tag{2.6}$$

For a fixed real h, these representations are defined by the operators

$$(R^h(\mathbf{h}(a, b, t + ab/2))f)(x) = e^{ih(t+ab/2)}e^{ihax}f(x + b) \tag{2.7}$$

and are irreducible only for $h \neq 0$. Due to the Stone-von Neumann theorem, for $h \neq 0$ representations given by (2.7) describe all the irreducible unitary representations of $H(3)$ whose restriction to the centre is nontrivial, up to unitary equivalence (see for example [14, 15]). Furthermore, there are characters of the form $\chi_{\alpha,\beta}(a, b, t) = e^{i(\alpha a+\beta b)}$ constituting representations of dimension one and being trivial on the center of $H(3)$.
Let us emphasize that the element $\mathbf{h}(0, 0, t)$ belongs to the group center Z of $H(3)$ and corresponds to the operator

$$R^h(\mathbf{h}(0, 0, t)) = e^{iht}I. \tag{2.8}$$

On the other hand the operator $R^h(\mathbf{h}(a, 0, 0))$ represents the multiplication by a function:

$$(R^h(\mathbf{h}(a, 0, 0))f)(x) = e^{ihax}f(x) \tag{2.9}$$

and $R^h(\mathbf{h}(0, b, 0))$ is the shift operator:

$$(R^h(\mathbf{h}(0, b, 0))f)(x) = f(x + b) \tag{2.10}$$

In representation R^h, it follows from the formulae (2.8)-(2.10) that the elements P, Q and E of $h(3)$ algebra correspond respectively to the operators P^h, Q^h and E^h given by

$$\begin{aligned} (P^h f)(x) &= \tfrac{d}{dx}f(x) \\ (Q^h f)(x) &= ihxf(x) \\ (E^h f)(x) &= ihf(x). \end{aligned} \tag{2.11}$$

and satisfy the same commutation relations as (2.5). It is possible to combine P^h and Q^h in one operator:

$$H^h = \{P^h\}^2 + \{Q^h\}^2 \tag{2.12}$$

which represents the Hamiltonian operator of harmonic oscillator algebra. The associated eigenfunctions $e_n^h(x)$ with eigenvalues $-\mu = -(2n+1)h$ read

$$e_n^h(x) = (2^n n!)^{-\frac{1}{2}}(\pi/h)^{-\frac{1}{4}} e^{-hx^2/2} H_n(\sqrt{h}x), \quad h > 0, \quad n = 0, 1, 2, ..., \tag{2.13}$$

and form an orthonormal basis of $L^2(\mathbb{R})$; H_n stand for Hermite polynomials. In terms of the differential notation (2.11) the eigenvalues equation $H^h y = -\mu y$ is nothing else than the harmonic oscillator differential equation

$$\frac{d^2y}{dx^2} + (\mu - h^2x^2)y = 0 \tag{2.14}$$

For our purpose, we use the quadratic transformation $t = x^2$ to put the later equation (2.14) in the form

$$t\frac{d^2y}{dt^2} + \frac{1}{2}\frac{dy}{dt} + \frac{1}{4}(\mu - h^2t)y = 0. \tag{2.15}$$

This equation admits two singularities: one in 0 which is elementary regular and one at ∞ which is unramified irregular; hence it can be characterized by the s-multisymbol $\{\frac{1}{2}; 2\}$.

3 The Mathieu differential equation and the motion group $M(2)$

In the canonical form, the Mathieu equation reads

$$\frac{d^2y}{ds^2} + (a - 2q\cos 2s)\, y = 0 \tag{3.1}$$

where s, q and the *characteristic value* a belong to $\mathbb{R}$. Among its solutions there are the *pseudo-periodic* ones (called Floquet solutions) of the form [8, 1]

$$me_\lambda(s,\ q) := e^{\lambda is} \sum_{k=-\infty}^{+\infty} C_k^\lambda e^{2kis} \tag{3.2}$$

where the coefficients C_k^λ satisfy the recursion relation

$$(a - (2k+\lambda)^2)C_k^\lambda - q(C_{k+2}^\lambda + C_{k-2}^\lambda) = 0. \tag{3.3}$$

Obviously we have

$$me_\lambda(s+\pi,\ q) = e^{\lambda\pi i} me_\lambda(s,\ q) \tag{3.4}$$

and

$$me_{-\lambda}(s,\ q) = me_\lambda(-s,\ q). \tag{3.5}$$

Of particular interest in physics and mathematics is the case $e^{\lambda\pi i} = \pm 1$, so that there exists at least one periodic solution of period π or 2π. In this case and when $q \to +\infty$ the characteristic value a can be approximated by

$$a = -2q + 2(2n+1)q^{1/2} - \frac{(2n+1)^2+1}{8} + O(n^3 q^{-1/2}),\ n \in \mathbb{N} \tag{3.6}$$

and for each q the periodic solution is either even or odd (often denoted by $ce_n(s,q)$ or $se_{n+1}(s,q)$). The change of variable $x = \cos^2 s$ in the Mathieu equation (3.1) leads to its algebraic form

$$x(1-x)\frac{d^2y}{dx^2} + \frac{1}{2}(1-2x)\frac{dy}{dx} + \frac{1}{4}(a+2q-4qx)y = 0 \tag{3.7}$$

which admits two elementary regular singular points at 0 and 1 and a ramified irregular singularity at infinity, so its s-multisymbol is $\{\frac{1}{2}; \frac{1}{2}; \frac{3}{2}\}$.

We will now consider the Lie algebra, denoted by $m_\alpha(2)$, generated by the three generators P_α, Q_α and E_α with the commutation relations [2]

$$[P_\alpha, Q_\alpha] = E_\alpha,\ \ [P_\alpha, E_\alpha] = -\alpha^2 Q_\alpha,\ \ [E_\alpha, Q_\alpha] = 0 \tag{3.8}$$

For $\alpha \neq 0$, this Lie algebra corresponds to the group $\tilde{G}_\alpha = \mathbb{R} \times \mathbb{R}^2$ equipped with the semi-direct product

$$(\theta, v) . (\theta', v') = (\theta + \theta', v + k_\alpha(\theta) . v') \tag{3.9}$$

where

$$k_\alpha(\theta) = \begin{pmatrix} \cos(\alpha\theta) & -\alpha \sin(\alpha\theta) \\ \alpha^{-1}\sin(\alpha\theta) & \cos(\alpha\theta) . \end{pmatrix} \tag{3.10}$$

The set of all $k_\alpha(\theta)$, for $\theta \in \mathbb{R}$, is denoted by $SO_\alpha(2)$. The group $\tilde{G}_\alpha$ is the simply connected covering of the group $M_\alpha(2) = SO_\alpha(2) \times \mathbb{R}^2 \equiv \mathbb{R}/_{2\pi\alpha^{-1}\mathbb{Z}} \times \mathbb{R}^2$ with the composition law

$$\left(\dot{\theta}, v\right) . \left(\dot{\theta}', v'\right) = \left(\theta \dot{+} \theta', v + k_\alpha(\theta) . v'\right) \tag{3.11}$$

where $\dot{\theta}$ designs the equivalence class of θ and $k_\alpha(\dot{\theta}) = k_\alpha(\theta)$. For $\alpha = 1$, $M_\alpha(2)$ is the euclidian motion group of the plane $M(2)$. For arbitrary α, $M_\alpha(2)$ is the group of displacements associated with the euclidean structure defined on $\mathbb{R}^2$ by

$$\| (v_1, v_2) \|_\alpha^2 = v_1^2 + \alpha^2 v_2^2. \tag{3.12}$$

This is why we call it the elliptic motion group of plane. For $\lambda \in \mathbb{R}/\mathbb{Z}$ we introduce the Hilbert space $\mathcal{H}^{\alpha,\lambda}$ of functions over $\mathbb{R}$ such that $f(\psi + 2\pi k\alpha^{-1}) = e^{2i\pi k\lambda} f(\psi)$ and which are square integrable over the pseudo-period $[0, 2\pi\alpha^{-1}]$. On $\mathcal{H}^{\alpha,\lambda}$ and for the real $h \neq 0$, we define an unitary irreducible representation of $\tilde{G}_\alpha$ by

$$\left(R_h^{\alpha,\lambda}(g) f\right)(\psi) = e^{ih\left(v_2 \cos(\alpha\psi) + \alpha^{-1} v_1 \sin(\alpha\psi)\right)} f(\psi + \theta). \tag{3.13}$$

where $g(\theta, v) = g(\theta, v_1, v_2) \in \tilde{G}_\alpha$ and $f \in \mathcal{H}^{\alpha,\lambda}$. This representation factorizes in a representation of $M_\alpha(2)$ if and only if f is $2\pi\alpha^{-1}$-periodic i.e. $\lambda = 0$. The infinitesimal operators are then:

$$\begin{aligned}
(P_h^{\alpha,\lambda} f)(\psi) := (R_h^{\alpha,\lambda}(P)f)(\psi) &= \frac{\partial}{\partial\theta}(R_h^{\alpha,\lambda}(g(\theta,0,0))f)(\psi)|_{\theta=0} = \frac{df}{d\psi}(\psi) \quad (3.14)\\
(Q_h^{\alpha,\lambda} f)(\psi) := (R_h^{\alpha,\lambda}(Q)f)(\psi) &= \frac{\partial}{\partial v_1}(R_h^{\alpha,\lambda}(g(0,v_1,0))f)(\psi)|_{v_1=0} = ih\alpha^{-1}\sin(\alpha\psi)f(\psi)\\
(E_h^{\alpha,\lambda} f)(\psi) := (R_h^{\alpha,\lambda}(E)f)(\psi) &= \frac{\partial}{\partial v_2}(R_h^{\alpha,\lambda}(g(0,0,v_2))f)(\psi)|_{v_2=0} = ih\cos(\alpha\psi)f(\psi)
\end{aligned}$$

We check easily that the operators $P_h^{\alpha,\lambda}$, $Q_h^{\alpha,\lambda}$ and $E_h^{\alpha,\lambda}$ verify the same relations as (3.8), and tend formally to the operators in (2.11) as $\alpha \to 0$.

On the other hand, we know that the motion group $M(2)$ is the symmetry group of the Helmholtz equation $[\triangle_2 + \omega]\psi(x,y) = 0$, where $\triangle_2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$. In [11], Miller had shown that the resolution of the above equation by the method of separation of variables could be realized in four orthogonal systems of coordinates: cartesian, polar, parabolic and elliptic associated respectively to four symmetric quadratic operators $L_1 = Q^2$, $L_2 = P^2$, $L_3 = \{P, Q\}$ and $L_4 = P^2 + d^2Q^2$ in the enveloping algebra of $M(2)$. Here P and Q are respectively the infinitesimal rotation and translation. L is called symmetric operator of Helmholtz equation if $[L, \mathtt{Q}] = R(x,y)\mathtt{Q}$ with $\mathtt{Q} = \triangle_2 + \omega$ and $R(x,y)$ is a complex function defined on $\mathbb{R}^2$. Corresponding to the representation of $M(2)$ every operator L_i corresponds naturally to a symmetric operator on a domain of definition D in $\mathcal{H}^{\alpha,0} \sim L^2(S^1)$ and each of them can be extended to a self-adjoint operator defined on a domain $D' \supseteq D$.

In the case of $\tilde{G}_\alpha$, where the above situation is very similar, and on the space of C^2 functions in $\mathcal{H}^{\alpha,\lambda}$ the elliptic operator $L_4^\alpha = \left\{P_h^{\alpha,\lambda}\right\}^2 + d^2\left\{Q_h^{\alpha,\lambda}\right\}^2$, associated to elliptic coordinate system, corresponds (for $d = 1$) to

$$L_4^\alpha = \frac{d^2}{d\psi^2} - h^2\alpha^{-2}\sin^2(\alpha\psi). \quad (3.15)$$

We have dropped the superscript λ, which does not appear in the right-hand side of equation (3.15). The operator L_4^α depends on it through the Hilbert space $\mathcal{H}^{\alpha,\lambda}$ on which it acts. The equation

$$L_4^\alpha y = -\mu y \quad (3.16)$$

after the change of variable $s = \alpha\psi + \frac{\pi}{2}$, is nothing else than the Mathieu equation with

$$a = \alpha^{-2}\mu - 2q \quad \text{and} \quad q = \frac{h^2\alpha^{-4}}{4}. \quad (3.17)$$

Setting $t = \alpha^{-2}\sin^2(\alpha\psi)$ in (3.16), we find the *deformed* algebraic form of the Mathieu equation

$$t(1-\alpha^2 t)\frac{d^2y}{dt^2} + \frac{1}{2}\{1 - 2\alpha^2 t\}\frac{dy}{dt} + \frac{1}{4}(\mu - h^2 t)y = 0 \quad (3.18)$$

This equation admits three singular points: 0, α^{-2} which are elementary regular and ∞ which is ramified irregular. Formally when $\alpha \to 0$ this equation tends to the harmonic oscillator differential equation (2.15). It is exactly this strong confluence that will be interpreted in terms of Lie algebra contraction in the next section.

4 The contraction of $M(2)$ on $H(3)$ and the confluence

Let consider the vector space V underlying to the Heisenberg algebra $h(3)$. It is generated by the basis P, Q and E. We denote by $[.,.]_0$ the Lie bracket:

$$[P,Q]_0 = E, \;\; [P,E]_0 = [E,Q]_0 = 0, \tag{4.1}$$

so that V endowed with this bracket $[.,.]_0$ is isomorphic to $h(3)$. We also denote by $[.,.]_1$ the Lie bracket defined by

$$[P,Q]_1 = E, \;\; [P,E]_1 = -Q, \;\; [E,Q]_1 = 0, \tag{4.2}$$

so that the vector space V equipped with $[.,.]_1$ is isomorphic to the Lie algebra $m(2)$. Let us introduce the following automorphism Φ_α of V:

$$\Phi_\alpha(P) = \alpha P, \quad \Phi_\alpha(Q) = \alpha Q, \quad \Phi_\alpha(E) = \alpha^2 E \tag{4.3}$$

and the Lie bracket $[.,.]_\alpha$ defined by

$$[X,Y]_\alpha = \Phi_\alpha^{-1}\left(\left[\Phi_\alpha(X), \Phi_\alpha(Y)\right]_1\right). \tag{4.4}$$

Then we have

$$[P,Q]_\alpha = E, \;\; [P,E]_\alpha = -\alpha^2 Q, \;\; [E,Q]_\alpha = 0. \tag{4.5}$$

It is obvious that V equipped with the Lie bracket $[.,.]_\alpha$ is isomorphic to $m_\alpha(2)$ and

$$\lim_{\alpha\to 0} [X,Y]_\alpha = [X,Y]_0 . \tag{4.6}$$

This means that the algebra $h(3)$ is a contraction of $m(2)$.
The authors of [2] have shown, using the orbits method, that when α goes to 0 the group $\tilde{G}_\alpha$ "tends" to a degree two extension G_0 of the Heisenberg group; and that the representation $R_h^{\alpha,\lambda}$, acting on $\mathcal{H}^{\alpha,\lambda}$, converges topologically (in the sense of Fell) to an irreducible representation of G_0 which restricts on $H(3)$ to the direct sum $R^h \oplus R^{-h}$ which acts on $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$. The geometric picture of this phenomenon is the following: the coadjoint orbits of $\tilde{G}_\alpha$ are cylinders of elliptic base, converging to the union of two planes of height $\pm h$ when $\alpha \to 0$, i.e. when the big axis of the ellipse grows to infinity. These two planes together form a coadjoint orbit of G_0.

To give a sense to the limit of representations, all the spaces $\mathcal{H}^{\alpha,\lambda}$ should be compared together and also with $L^2(\mathbb{R}) \oplus L^2(\mathbb{R})$, using the multiresolution analysis according to

Mallat and Meyer [7, 9, 3], see appendix 1.

Finally the confluence holds when $\alpha \to 0$; the elliptic operator L_4^α tends formally to the Hamiltonian operator H^h of oscillator harmonic algebra, and in virtue of (3.6) and (3.17) μ tends to $(2n+1)h$ and the equation (3.18) becomes the equation (2.15). At the same time the differential equation solutions experience the following limits (recall that h is a fixed positive parameter, and that q and α are related by the equality $q = \frac{1}{4}h^2\alpha^{-4}$):

$$\begin{aligned}
\lim_{\alpha\to 0} \frac{ce_{2n}(\alpha\psi + \frac{\pi}{2}, q)}{ce_{2n}(\frac{\pi}{2}, q)} &= \frac{\Gamma(\frac{1}{2}-n)}{2^n \pi^{\frac{1}{2}}}\, e^{-h\psi^2/2} H_{2n}(\sqrt{h}\psi) \\
&= \frac{(-1)^n 2^n n!}{(2n)!} e^{-h\psi^2/2} H_{2n}(\sqrt{h}\psi) \\
\lim_{\alpha\to 0} \frac{se_{2n+2}(\alpha\psi + \frac{\pi}{2}, q)}{se'_{2n+2}(\frac{\pi}{2}, q)} &= \frac{\Gamma(-\frac{1}{2}-n)}{2^{n+1} \pi^{\frac{1}{2}}}\, e^{-h\psi^2/2} H_{2n+1}(\sqrt{h}\psi) \\
&= \frac{(-1)^{n+1} 2^{n+1} (n+1)!}{(2n+2)!} e^{-h\psi^2/2} H_{2n+1}(\sqrt{h}\psi). \qquad (4.7)
\end{aligned}$$

In fact and according to [8] (Satz 10 paragraph 2.333), we have in the interval $[0, \pi]$

$$\left.\begin{array}{r} ce_n(z,q) \\ se_{n+1}(z,q) \end{array}\right\} = \left(\frac{\pi q^{\frac{1}{2}}}{2}\right)^{\frac{1}{4}} (n!)^{-\frac{1}{2}} D_n(2q^{\frac{1}{4}}\cos z) + O(q^{-\frac{3}{8}}) \qquad (4.8)$$

as $q \to \infty$, where $D_m(\zeta)$ is the parabolic cylinder function given by

$$D_m(\zeta) = \frac{1}{2^{m/2}} e^{-\frac{\zeta^2}{4}} H_m\left(\frac{\zeta}{\sqrt{2}}\right).$$

5 Conclusion and discussion

In this work we have examined the singularities confluence of the Mathieu differential equation towards the harmonic oscillator differential equation via the Lie algebra contraction of the motion Lie algebra $m(2)$ to the Heisenberg Lie algebra $h(3)$. The use of the contraction method to interpret successfully the confluence was based on the approach of the work [2]. Someone can now be tempted to develop similar interpretations of singularities confluence for other differential equations. Among many examples we can cite the case of Lamé differential equation with four regular singularities 0, 1, a (which are elementary) and ∞ associated with the s-multisymbol $\{\frac{1}{2}; \frac{1}{2}; \frac{1}{2}; 1\}$

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left[\frac{1}{x} + \frac{1}{x-1} + \frac{1}{x-a}\right]\frac{dy}{dx} + \frac{\mu - l(l+1)x}{4x(x-1)(x-a)} y = 0 \qquad (5.9)$$

If $a = r^{-2}$ and $x = sn^2(z, r)$, where $sn(z, r)$ is the Jacobi elliptic function, this equation becomes the Lamé differential equation in the jacobian form

$$\frac{d^2y}{dz^2} - r^2\left\{l(l+1)sn^2(z,r) - \mu\right\} y = 0 \qquad (5.10)$$

where we have used the fact

$$\frac{d}{dz}sn(z,r) = cn(z,r)dn(z,r) = \sqrt{(1-sn^2(z,r))(1-r^2sn^2(z,r))} \tag{5.11}$$

and so

$$\frac{dx}{dz} = 2sn(z,r)cn(z,r)dn(z,r) = 2\sqrt{x(1-x)(1-r^2x)} \tag{5.12}$$

The equation (5.9), which is a special case of Heun differential equation, is related to the group $SO_0(2,1)$. In fact it was shown in [16] that the resolution by separation of variables of the Laplacian equation $\mathtt{Q}f = l(l+1)f$ on the hyperboloid $x_0^2 - x_1^2 - x_2^2 = 1$, where $\mathtt{Q} = K_1^2+K_2^2-M_3^2$ and $K_1 = -x_0\partial_{x_2}-x_2\partial_{x_0}$, $K_2 = -x_0\partial_{x_1}-x_1\partial_{x_0}$ and $M_3 = x_1\partial_{x_2}-x_2\partial_{x_1}$, can be realized in nine coordinate systems associated with nine symmetric quadratic operators in the enveloping algebra of $SO_0(2,1)$. The above operators correspond to symmetric operators on the domain D of C^∞ functions in $\mathcal{H} = L^2(S^1)$ corresponding to the principal series representations of $SO_0(2,1)$ and each operator can be extended to one or more self-adjoint operators on $\mathcal{H}$ (see ref. [5]). Of special interest is the elliptic operator $L_E = M_3^2 + k^2K_2^2$ ($k \in \mathbb{R}$) associated to the elliptic coordinate system. It corresponds on the Hilbert space $\mathcal{H} = L^2(S^1)$ to

$$L_E = (1+k^2\cos^2\theta)\frac{d^2}{d\theta^2} + k^2(2l-1)\sin\theta\cos\theta\frac{d}{d\theta} + k^2(l^2\sin^2\theta + l\cos^2\theta)$$

corresponding to the principal series ($l = -\frac{1}{2}+i\rho, \quad 0<\rho<\infty$). To retrieve Lamé equation (5.9) we use the variable change

$$f(\theta) = (1+k^2\cos^2\theta)^{l/2}y(x), \tag{5.13}$$

and $x = \frac{\sin^2\theta}{1+k^2\cos^2\theta}$, $a = -\frac{1}{k^2}$ in the equation $L_Ef = \mu k^2 f$.

On the other hand, the contraction of the group $SO_0(2,1)$ on the Heisenberg group $H(3)$ brings the possibility to interpret the double confluence of the Lamé differential equation to harmonic oscillator differential equation. In fact, we set

$$P^\alpha = \alpha M_3, \quad Q^\alpha = -\alpha^2 K_2, \text{ and } E = \alpha^3 K_1 \tag{5.14}$$

then we have

$$[P^\alpha, Q^\alpha] = E^\alpha, \quad [E^\alpha, P^\alpha] = \alpha^2 Q^\alpha, \quad [E^\alpha, Q^\alpha] = \alpha^4 P^\alpha \tag{5.15}$$

and in the limit $\alpha \to 0$ we obtain the commutation relations (2.5) of the Heisenberg Lie algebra. Similarly as above, setting $f(\theta) = (1+\alpha^2k^2\cos^2\alpha\theta)^{l/2}y(x)$ and $x = \frac{\alpha^{-2}\sin^2\alpha\theta}{1+\alpha^2k^2\cos^2\alpha\theta}$ in the equation $L_E^\alpha f = -\mu f$, where $L_E^\alpha = \{P^\alpha\}^2 + k^2\{Q^\alpha\}^2$ ($k \in \mathbb{R}$) is the deformed elliptic operator associated to the elliptic coordinate system and corresponding to the principal series on D in the Hilbert space $\mathcal{H} = L^2(S^1)$, we obtain the deformed Lamé equation

$$\frac{d^2y}{dx^2} + \frac{1}{2}\left[\frac{1}{x} + \frac{\alpha^2}{\alpha^2x-1} + \frac{\alpha^4k^2}{\alpha^4k^2x+1}\right]\frac{dy}{dx} + \frac{-\mu-\alpha^6k^2l(l+1)x}{4x(\alpha^2x-1)(\alpha^4k^2x+1)}y = 0 \tag{5.16}$$

In the limit $\alpha \to 0$, after setting $\rho = \alpha^{-3}k^{-1}h$, this equation becomes the harmonic oscillator differential equation (2.15). In this process, the two regular singularities 1 and a coalesce to ∞ and this double confluence is also strong. Furthermore the contraction of $SO_0(2,1)$ on the motion group $M(2)$ [10] permits us to interpret the confluence of the Lamé differential equation to the Mathieu differential equation. In this case the elementary regular singular point a coalesces to ∞. Moreover, the result that the periodic solutions of the Lamé equation tend to periodic solutions of Mathieu equation already exists in the work of Kalnins *et al.* [6] given in terms of the contraction of $SO(3)$ to $M(2)$. Therein, the irreducible representation of $SO(3)$ is labeled by l (integer) and the periodic Lamé solutions are polynomials which experience the following limits

$$\lim_{l\to+\infty}\prod_{j=1}^{2m}(1-\frac{u}{\theta_j}) = \frac{ce_{2m}(\theta,q)}{ce_{2m}(0,q)}, \quad \lim_{l\to+\infty} u^{\frac{1}{2}}\prod_{j=1}^{2m+1}(1-\frac{u}{\theta_j}) = \frac{se_{2m+1}(\theta,q)}{se'_{2m+1}(0,q)} \tag{5.17}$$

$$\lim_{l\to+\infty}(1-u)^{\frac{1}{2}}\prod_{j=1}^{2m+1}(1-\frac{u}{\theta_j}) = \frac{ce_{2m+1}(\theta,q)}{ce_{2m+1}(0,q)}, \quad \lim_{l\to+\infty} u^{\frac{1}{2}}(1-u)^{\frac{1}{2}}\prod_{j=1}^{2m+2}(1-\frac{u}{\theta_j}) = \frac{se_{2m+2}(\theta,q)}{se'_{2m+2}(0,q)}$$

where $u = \sin^2\theta$, $q = \frac{h^2}{4}$, $a = \frac{l^2}{h^2}$ and the zeros θ_j satisfy some fundamental relation which depends on a. Finally we can summarize this study by the following commutative diagram which exhibits the interpretation of confluence phenomena by the Lie algebra contraction:

$$\begin{tikzcd}
\text{Lamé D.E.} \arrow[dd, "\text{Confluence}"'] \arrow[dr] & & & SO_0(2,1) \arrow[lll, Rightarrow] \arrow[dl] \arrow[dd, "\text{Contraction}"] \\
& \text{Mathieu D.E.} \arrow[dl] & M(2) \arrow[l, Rightarrow] \arrow[dr] & \\
\text{H.O.D.E} & & & H(3) \arrow[lll, Rightarrow]
\end{tikzcd}$$

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Appendix 1
Multiresolution analysis

In this appendix we recall some background on Multiresolution analysis. So it is useful in below to identify $\mathcal{H}^{\alpha,\lambda}$ space with $\mathcal{V}^{\alpha,\lambda} = \ell^2(\alpha(\mathbb{Z}+\lambda))$ space by the isometric isomorphism $\mathcal{F} : f \mapsto \hat{f}$ defined, for $x \in \alpha(\mathbb{Z}+\lambda)$, by

$$\hat{f}(x) = \int_{\mathbb{R}/2\pi\alpha^{-1}\mathbb{Z}} e^{-ix\psi} f(\psi) d\psi. \tag{1.18}$$

We recall briefly the construction of Littlewood-Paley-Meyer (LPM) wavelets [2, 9]. Let ϕ a function of class C^∞ such that ¶

$$\begin{array}{rl} \phi(\xi) = 1 & \forall\ |\xi| \leq \frac{2\pi}{3}, \\ 0 < \phi(\xi) < 1 & \forall\ \frac{2\pi}{3} < |\xi| \leq \frac{4\pi}{3}, \\ \phi(\xi) = 0 & \forall\ |\xi| \geq \frac{4\pi}{3}, \\ \phi^2(\xi) + \phi^2(2\pi - \xi) = 1 & \forall\ 0 \leq \xi \leq 2\pi, \end{array} \tag{1.19}$$

We denote by $V^{\alpha,\lambda}$ the subspace of $L^2(\mathbb{R})$ of functions having the Fourier transformation of the form

$$m(\xi)\phi(\alpha\xi) \tag{1.20}$$

where $m \in \mathcal{H}^{\alpha,\lambda}$.
Using the proprieties of function ϕ, we remark that

$$\sum_{k\in\mathbb{Z}} \phi^2(\xi + 2k\pi) = 1 \tag{1.21}$$

which leads to

$$\int_{-\infty}^{+\infty} m_1(\xi)\overline{m_2(\xi)}\phi^2(\alpha\xi)d\xi = \int_0^{2\pi\alpha^{-1}} m_1(\xi)\overline{m_2(\xi)}d\xi \tag{1.22}$$

Then the functions

$$\phi_k^\lambda(\xi) = \alpha^{\frac{1}{2}} e^{i(k+\lambda)\alpha\xi}\phi(\alpha\xi), \quad k \in \mathbb{Z} \tag{1.23}$$

form an orthonormal basis for the Fourier transform $\mathcal{F}V^{\alpha,\lambda}$ of $V^{\alpha,\lambda}$ space. And hence the functions $\hat{\phi}_k^\lambda(x) = \alpha^{-\frac{1}{2}}\hat{\phi}(\alpha^{-1}(x-k-\lambda)$ form an orthonormal basis of $V^{\alpha,\lambda}$, which give us a family of isometric injections:

$$I_{\alpha,\lambda} : \mathcal{V}^{\alpha,\lambda} \tilde{\to} V^{\alpha,\lambda} \subset L^2(\mathbb{R}) \tag{1.24}$$

¶As an example of ϕ function we can adopt the one of [17]:

$$\phi(\psi) = \sqrt{g(\psi)g(-\psi)}$$

where $g(\psi) = \frac{h(4\pi/3-\psi)}{h(\psi-2\pi/3)+h(4\pi/3-\psi)}$, $h(\psi) = \exp(-1/\psi^2)$, $\psi > 0$

The Meyer proposition stipulates that for all $\alpha > 0$, and $\lambda \in \mathbb{R}/\mathbb{Z}$ we have:

1. $V^{\alpha,\lambda} \subset V^{\alpha',\lambda'}$ when $\alpha(\mathbb{Z}+\lambda) \subset \alpha'(\mathbb{Z}+\lambda')$
2. $\bigcap_{n\in\mathbb{Z}} V^{2^n\alpha,2^{-n}\lambda} = \{0\}$, and $\overline{\bigcup_{n\in\mathbb{Z}} V^{2^n\alpha,2^{-n}\lambda}} = L^2(\mathbb{R})$,
3. For all $a > 0$, $f(x) \in V^{\alpha,\lambda} \Leftrightarrow a^{-1/2} f(a^{-1}x) \in V^{a\alpha,\lambda}$

and for all $b \in \alpha\mathbb{Z}$, if $f(x) \in V^{\alpha,\lambda}$ then $f(x-b) \in V^{\alpha,\lambda+b/\alpha}$ (i.e. the injections $I_{\alpha,\lambda}$ commute with dilations and translations).

The family of the isometric injections $I_{\alpha,\lambda}$ is called *multiresolution analysis* of $L^2(\mathbb{R})$ and is ∞-*regular* [‖]. It furnishes a precise meaning to the intuitive fact that $\mathcal{H}^{\alpha,\lambda}$ "tends" to $L^2(\mathbb{R})$ when α goes to 0. To construct the isometric injections on two copies of $L^2(\mathbb{R})$ it is enough to identify in natural way the space $\mathcal{H}^{\alpha,\lambda}$ to two copies of $\mathcal{H}^{2\alpha,\lambda/2}$. In fact considering the operators

$$\mathcal{U}_{\alpha,\lambda} : \mathcal{H}^{\alpha,\lambda} \to \mathcal{H}^{\alpha,\lambda}, \qquad \mathcal{J} : \mathcal{H}^{2\alpha,\lambda/2} \to \mathcal{H}^{\alpha,\lambda}$$

defined by

$$\mathcal{U}_{\alpha,\lambda}.f(\psi) = e^{-i\pi\lambda} f\left(\psi + \pi\alpha^{-1}\right), \qquad \mathcal{J}.f(\psi) = \left(1 + e^{i\alpha\psi}\right) f(\psi). \tag{1.25}$$

The application

$$(\mathcal{J},\, \mathcal{U}_{\alpha,\lambda} \circ \mathcal{J}) : \mathcal{H}^{2\alpha,\lambda/2} \oplus \mathcal{H}^{2\alpha,,\lambda/2} \to \mathcal{H}^{\alpha,\lambda} \tag{1.26}$$

is an isometric isomorphism with inverse given by

$$\begin{pmatrix} \mathcal{R} \\ \mathcal{R} \circ \mathcal{U}_{\alpha,\lambda} \end{pmatrix} \tag{1.27}$$

where $\mathcal{R}$ is the adjoint of the injection $\mathcal{J}$ and reads

$$\mathcal{R}.f(\psi) = \frac{1}{4}\left(1 + e^{-i\alpha\psi}\right) f(\psi) + \frac{1}{4}\left(1 - e^{-i\alpha\psi}\right) f\left(\psi + \pi\alpha^{-1}\right). \tag{1.28}$$

Now let $I_{\alpha,\lambda}$ a multiresolution analysis of Littlewood-Paley-Meyer of $L^2(\mathbb{R})$, so for all $f \in \mathcal{H}^{\alpha,\lambda}$ we define $\mathcal{I}_{\alpha,\lambda}$ by:

$$\mathcal{I}_{\alpha,\lambda}(f)(\psi) := \mathcal{F}^{-1} \circ I_{\alpha,\lambda} \circ \mathcal{F}(f)(\psi) = \phi(\alpha\psi) f(\psi) \tag{1.29}$$

If we set, for any $u \in L^2(\mathbb{R})$,

$$\mathcal{A}_{\alpha,\lambda} u(\psi) = \sum_{k\in\mathbb{Z}} \phi(\alpha\psi + 2k\pi) e^{-2ik\pi\lambda} u(\psi + 2k\pi\alpha^{-1}) \tag{1.30}$$

then $\mathcal{A}_{\alpha,\lambda} u \in \mathcal{H}^{\alpha,\lambda}$. By the virtue of (1.21) we have, for any $f \in \mathcal{H}^{\alpha,\lambda}$, $(\mathcal{A}_{\alpha,\lambda} \circ \mathcal{I}_{\alpha,\lambda}) f(\psi) = f(\psi)$.

[‖] A multiresolution analysis is said ∞-*regular*, if any Dirac mass is sent by $I_{\alpha,\lambda}$ on a function of the Schwartz space $\mathcal{S}(\mathbb{R})$.

We can check easily that $\mathcal{A}_{\alpha,\lambda}$ is the adjoint of $\mathcal{I}_{\alpha,\lambda}$ and that $\mathcal{P}_{\alpha,\lambda} = \mathcal{I}_{\alpha,\lambda} \circ \mathcal{A}_{\alpha,\lambda}$ is an orthogonal projector of $L^2(\mathbb{R})$ on the Fourier transform $\mathcal{F}V^{\alpha,\lambda}$ of the space $V^{\alpha,\lambda}$. And then for all $u,\ v \in \mathcal{S}(\mathbb{R})$ and $X \in V$, the following expression

$$\begin{pmatrix} \mathcal{I}_{2\alpha,\lambda/2} & 0 \\ 0 & \mathcal{I}_{2\alpha,\lambda/2} \end{pmatrix} \begin{pmatrix} \mathcal{R} \\ \mathcal{R} \circ \mathcal{U}_{\alpha,\lambda} \end{pmatrix} \circ R_h^{\alpha,\lambda}(exp_\alpha X) \circ (\mathcal{J},\ \mathcal{U}_{\alpha,\lambda} \circ \mathcal{J}) \begin{pmatrix} \mathcal{A}_{2\alpha,\lambda/2} & 0 \\ 0 & \mathcal{A}_{2\alpha,\lambda/2} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.31}$$

tends to

$$\begin{pmatrix} R^h(exp_0 X) & 0 \\ 0 & R^{-h}(exp_0 X) \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \tag{1.32}$$

when α tends to 0, the convergence holding in $\mathcal{S}(\mathbb{R}) \oplus \mathcal{S}(\mathbb{R})$ in Fréchet sense and uniformally for all X belong to a compact set of V.

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